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CRLB and ML for parametric estimate: new results

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Abstract.

In the present paper, we calculate the Cramér-Rao Lower Bound (CRLB) for the P_d<1 and P_{fa}>0 case, accounting both for missed detections and false alarms, for an estimation problem in a surveillance system where the measurements are acquired by a radar. The analysis, which is limited to the parameter estimation for a deterministic steady-state target motion, is a further innovation in the computation of the CRLB of ML estimators for deterministic steady–state models; in fact, the classical theory applies only in the unrealistic case of P_d=1 and P_{fa}=0. The results obtained by means of a Monte Carlo simulation successfully validate our extended enhanced estimation method.

Keywords: MLE, CRLB, ballistic target model, detection probability, probability of false alarm, target tracking.

List of acronyms

CRLB	Cramér-Rao Lower Bound
FIM	Fisher Information Matrix
IRF	Information Reduction Factor
ML	Maximum Likelihood
MLE	Maximum Likelihood Estimator

1 Introduction

The Cramér-Rao Lower Bound (CRLB) plays an important role in the estimation theory because it theoretically predicts the best achievable second-order estimation error performance [1]. Traditionally, the main source of uncertainty in the parameter estimation and the derivation of the corresponding CRLB is the measurement noise. In many engineering problems (e.g. target tracking), due to the imperfections in target detection, there is an additional source of uncertainty in estimation, caused by the unknown measurement origin (target detections or false alarm occurrences). In other words, the assumption of detection probability P_d=1 and false alarm probability P_{fa}=0 is unrealistic in these applications; this consideration applies also when the Maximum Likelihood Estimator (MLE) is determined and the CRLB of the same is evaluated.

The effect of uncertain measurements has been first observed in [2] and [3], with respect to a deterministic target motion: it is shown that the FIM is scaled by a constant factor less than unity, the so-called Information Reduction Factor (IRF). In mathematical terms, these results are synthesized via the equation (see also Eq. (32) of [2] and Eq. (52) of [3]):

$$\begin{aligned} & FIM\left(P_d < 1, P_{fa} > 0\right) = \\ & \cong FIM\left(P_d = 1, P_{fa} = 0\right) \cdot IRF\left(P_d, P_{fa}\right) \end{aligned} \tag{1.1}$$

The bound resulting from Eq. (1.1) is effective in modeling the measurement origin uncertainty as the product of two distinct factors: the FIM relative to the ideal case (P_d = 1, P_{fa} = 0) and the IRF accounting for missed detections and false alarms.

The CRLB reported in [4] is a generalization allowing for non linear measurements and non deterministic dynamics; also in this extended context, the measurement origin uncertainty can be expressed by means of a suitable IRF.

In [5], the theoretical CRLB for P_d < 1 and P_{fa} = 0 (i.e., in absence of false alarms) has been calculated via the enumeration of all possible sequences of detections and missed detections, given a certain scan number. The computational complexity of this solution grows exponentially with time, but the bound has been verified to be the exact one, by means of numerical simulations, both in linear and non linear cases. An approximation of the theoretical bound for practical applications, where the number of sensor scans is large, has also been proposed.

The comparison of the two bounds (IRF and enumeration method) has been reported in [6], showing, both theoretically and with numerical examples, that the CRLB computed via the IRF is overly optimistic. The enumeration based CRLB is the true bound for P_d<1 and P_{fa}=0 case, since all possible sequences of detections and missed detections are averaged.
In the present paper, we extend the enumeration method for CRLB computation, to the parameter estimation problem with P_d<1 and P_{fa}>0. A case study, which

compares the MLE with this new CRLB, is presented in the context of a ballistic target tracking with a radar. The rationale of the work rests on the features of ML estimators, the most suitable for parametric problems, and on the yardstick for the second-order error performance that the CRLB analysis provides.

The paper is organized as follows:

- Section 2 presents the mathematical formulation of the enumeration method, illustrating all possible sequences of detections and missed detections for true and false targets. The theoretical CRLB for non-linear parameter estimation is calculated with $P_d<1$ and $P_{fa}>0$.
- Section 3 presents a case study. It describes the application of the enumeration method, with $P_d<1$ and $P_{fa}>0$, to the estimation of the impact point of a ballistic target, e.g. an artillery shell.
- Section 4 illustrates the Monte Carlo simulation and compares its results to the CRLB.
- Conclusions are reported in Section 5.

2 Enumeration method

2.1 Background

This section extends the enumeration method illustrated in [5] to all the possible sequences of detections and missed detections in presence of true and false targets. The method proposed in the following relies on the assumption of constant measurement accuracy.

Assume we have a radar that acquires measurements on the target position. Let us consider a discrete-time deterministic target motion. The measurement equation when a target is detected is given by:

$$\mathbf{z}_k = \mathbf{h}_k\left(\mathbf{x}_k\right) + \mathbf{v}_k \tag{2.1}$$

where k is the discrete-time index (or scan number), $\mathbf{x}_k \in \Re^n$ is the target state vector, $\mathbf{z}_k \in \Re^m$ is the measurement at k and $\mathbf{v}_k$ is the zero-mean white Gaussian measurement noise with non-singular covariance matrix $\mathbf{R}_k$.

The events of target detection and target missed detection at time k are described by the binary variable $d_{T;k}$:

$$d_{T;k} = \begin{cases} 1, & \text{if target was detected} \\ 0, & \text{if target was missed.} \end{cases} \tag{2.2}$$

Eq. (2.1) can be applied also in the event of target missed detection, by defining the inverse of the covariance matrix of measurement noise as $d_{T;k} \cdot \mathbf{R}_k^{-1}$. In other words, when the target is missed, the covariance of the measurement is set to ∞, or, in practice, to a very large value[1]. We assume there can be only one target originated detection in the acquisition window, because we hypothesize a suitable sidelobe blanking device is installed and used within the radar, thus excluding sidelobe targets.

The presence of false alarms is managed according to the following hypotheses:

- There cannot be more than one false alarm per scan in the radar acquisition window. This assumption actually depends on the radar sensor and on the size of the acquisition window, but it is justified in most cases.
- The false alarm probability per scan is P_F, generally different from the sensor P_{fa}. P_F differs from P_{fa} for a number of reasons; e.g.: many false plots, generated for instance by decoys, debris, junk, are likely to be in the acquisition window and the plurality of false alarms might be summarized by just one false alarm with higher probability. The hypothesis of having just one false alarm will allow us to avoid the cumbersome, though exact, mathematics to derive the MLE and the CRLB as reported in [6].

For false alarms, the following measurement equation is adopted:

$$\mathbf{z}_k = \mathbf{w}_k + \mathbf{v}_k \tag{2.3}$$

where $\mathbf{w}_k$ is a suitable random variable uniformly distributed in the radar acquisition window and $\mathbf{v}_k$ is the zero-mean white Gaussian measurement noise with non-singular covariance matrix $\mathbf{R}_k$. Also in this case, the application of the measurement equation in the event no false alarm occurrences is obtained by setting the covariance of the measurement to a very large value.

Given the k^{th} scan, two statistically independent events are simultaneously possible: target detection, with probability P_d and binary value $d_{T;k}$, and false alarm occurrence, with probability P_F and binary value $d_{F;k}$. The binary variable $d_{F;k}$ describes false alarm occurrences:

$$d_{F;k} = \begin{cases} 1, & \text{if false alarm occured} \\ 0, & \text{if false alarm did not occur.} \end{cases} \tag{2.4}$$

At time index k, 2^k detection and missed detection sequences are possible for the target, and similarly 2^k occurrence and absence sequences are possible for the false alarm. Respectively, such sequences are as follows [1]:

[1] The mathematical expedient of assigning a very large value to the measurement variance is done for convenience only (i.e., a way to keep the measurement out of the CRLB computation).

$$\begin{cases} S_{T;k,l} : d_{T;1,l} \quad d_{T;2,l} \ \ldots \ d_{T;k,l} \\ S_{F;k,m} : d_{F;1,m} \ d_{F;2,m} \ \ldots \ d_{F;k,m} \end{cases} \qquad l, m = 1,\ldots,2^k \tag{2.5}$$

The number of detections of the target in a particular sequence $S_{T;k,l}$ and the number of occurrences of false alarms in a particular sequence $S_{F;k,m}$ are respectively given by [1]:

$$\begin{cases} \Delta_{T;k,l} = \sum_{i=1}^{k} d_{T;i,l} \\ \Delta_{F;k,m} = \sum_{i=1}^{k} d_{F;i,m} \end{cases} \tag{2.6}$$

For a certain detection probability $P_d<1$, which is taken constant over the scan number k, the probability of occurrence of a particular sequence of detections and missed detections of the target $S_{T;k,l}$ is [5]:

$$\Pr\{S_{T;k,l}\} = P_d^{\Delta_{T;k,l}} \cdot (1-P_d)^{k-\Delta_{T;k,l}} \tag{2.7}$$

For a given false alarm per scan probability P_F, corresponding to a false alarm probability $P_{fa}>0$, the probability of occurrence of a particular sequence of detections and missed detections of false targets $S_{F;k,m}$ is:

$$\Pr\{S_{F;k,m}\} = P_F^{\Delta_{F;k,m}} \cdot (1-P_F)^{k-\Delta_{F;k,m}} \tag{2.8}$$

At time index k, each of the 2^k sequences relative to the target may be combined with the 2^k sequences associated to false alarms. The sequences resulting from such combination are 2^{2k} and, due to the statistical independence of true and false target detections, their probability of occurrence is obtained by multiplication of (2.7) and (2.8), for each possible pair (l, m) in Eq. (2.5):

$$\Pr\{S_{k;l,m}\} = \Pr\{S_{T;k,l}\} \cdot \Pr\{S_{F;k,m}\} \qquad l, m = 1,\ldots,2^k \tag{2.9}$$

where the subscript $k;l,m$ means that, at the k^{th} scan, we consider the sequence l^{th} for the target and m^{th} for the false alarm.

2.2 Theoretical Cramér-Rao Lower Bound

Let us consider a certain sequence $S_{T;k,l}$ of detections and missed detections of the target; the covariance of an unbiased target state estimator is bounded as follows [5]:

$$E\left\{(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T \mid S_{T;k,l}\right\} \geq \left[\mathbf{J}_{T;k}\left(S_{T;k,l}\right)\right]^{-1} \tag{2.10}$$

The matrix $\mathbf{J}$ is the FIM, so the limit on the right-hand side of (2.10) is the CRLB, conditioned upon the particular sequence of detections and missed detections of the target:

$$\mathbf{CRLB}_{T;k,l} = \left[\mathbf{J}_{T;k,l}\left(S_{T;k,l}\right)\right]^{-1} \tag{2.11}$$

False alarms are characterized by their own CRLB. In particular, given a certain sequence $S_{F;k,m}$ at scan *k*, we assume the false alarm conditional CRLB is related to the target CRLB, according to the following expression:

$$\mathbf{CRLB}_{F;k,m} = f \cdot \mathbf{CRLB}_{T;k,m} \qquad m = 1,\ldots,2^k \tag{2.12}$$

where f is a suitable parameter greater than unity. Eq. (2.12) comes from the assumption that false alarms are uniformly distributed in a search area wider than the radar resolution cell and it accounts both for radar measurements uncertainty and false alarms intrinsic random nature. In particular, for the radar acquisition window typically being three times the radar measurement error, the coefficient f equals 9, since the CRLB accounts for the error variance.

In the case under investigation ($P_d<1$ and $P_{fa}>0$), the possible events are a combination of target detections and false alarm occurrences. At time index *k*, an unbiased target state estimator is still bounded by the CRLB, but its computation must take into account the existence both of target and false alarms, suitably weighting their contributions.

Each sequence $S_{k;l,m}$ corresponds to a particular combination of target detections and false alarm occurrences. The bound $\mathbf{CRLB}_{k;l,m}$ is conditioned on the sequence pair (l, m), whose probability is expressed by Eq. (2.9). It depends on the target CRLB at time instant *k* and for sequence *l* and on the false alarm CRLB at time instant *k* and for sequence *m*. In particular, three contributions need to be considered:

1. the target CRLB is weighted by the ratio of the number of target detections $\Delta_{T;k,l}$ (2.6) in the sequence $S_{T;k,l}$ (2.5) over the total number of scans *k*;
2. the false alarm CRLB is weighted by the ratio of the number of false alarm occurrences $\Delta_{F;k,m}$ (2.6) in the sequence $S_{F;k,m}$ (2.5) over the total number of scans *k*;
3. given the sequences $S_{T;k,l}$ and $S_{F;k,m}$, there might be a certain number of scans $\Omega_{k;l,m}$ in which the target is detected and the false alarm simultaneously occurs:

$$\Omega_{k;l,m} = \sum_{i=1}^{k} d_{T;i,l} \cdot d_{F;i,m} \tag{2.13}$$

If $\Omega_{k;l,m} \neq 0$, then the target and the false alarm contributions to the overall CRLB must be both reduced, proportionally to the ratios $\frac{\Omega_{k;l,m}}{\Delta_{T;k,l}}$ and $\frac{\Omega_{k;l,m}}{\Delta_{F;k,m}}$.

A further adjustment is needed, when $\Omega_{k;l,m} \neq 0$, to take into consideration that the effective number of (true and false) target detections is $\Delta_{T;k,l} + \Delta_{F;k,m} - \Omega_{k;l,m}$.

Thus, the ultimate expression for the bound $\mathbf{CRLB}_{k;l,m}$ is:

$$\mathbf{CRLB}_{k;l,m} = \\ = \left\{ \begin{array}{l} \frac{\Delta_{T;k,l}}{k} \mathbf{CRLB}_{T;k,l} + \\ + \frac{\Delta_{F;k,m}}{k} \mathbf{CRLB}_{F;k,m} + \\ - \left(\frac{1}{2}\right)^2 \cdot \Omega_{k;l,m} \cdot \left(\frac{\mathbf{CRLB}_{T;k,l}}{\Delta_{T;k,l}} + \frac{\mathbf{CRLB}_{F;k,m}}{\Delta_{F;k,m}} \right) \end{array} \right\} \times \\ \times \frac{\Delta_{T;k,l} + \Delta_{F;k,m} - \Omega_{k;l,m}}{\Delta_{T;k,l} + \Delta_{F;k,m}} \tag{2.14}$$

being $\Delta_{T;k,l}$ and $\Delta_{F;k,m}$ given by (2.6), $\mathbf{CRLB}_{T;k,l}$ and $\mathbf{CRLB}_{F;k,m}$ respectively by (2.11) and (2.12) and $\Omega_{k;l,m}$ by (2.13).

The first two terms in Eq. (2.14) respectively represent the target and the false alarm contributions when they do not interact with each other, i.e. when there is no scan in which the radar simultaneously reveals them. The third term accounts for the fact that, when the target is detected and the false alarm simultaneously occurs, we suppose to average the recorded measurements before processing them. So, $\Omega_{k;l,m}$ times out of $\Delta_{T;k,l}$ for the target and $\Omega_{k;l,m}$ times out of $\Delta_{F;k,m}$ for the false alarm, the target and the false alarm CRLB matrices have to be reduced by the factor 0.5^2 (if measurements are straight averaged, then variances need to be multiplied by the square of $1/2)^2$. Moreover, in case of simultaneous target detection and false alarm occurrence, the effective number of (true and false) target detections is overstated, thus imposing the correction embodied by the last multiplying ratio, which is equal to 1 in case there are no simultaneous detections (i.e., $\Omega_{k;l,m} = 0$).

The bound in (2.14) is conditioned upon a particular sequence $S_{k;l,m}$ of target detections and false alarm occurrences. The unconditional CRLB is obtained by taking the expectation of (2.14) with respect to $S_{k;l,m}$; the theoretical CRLB with $P_d<1$ and $P_{fa}>0$ can be written as:

$$\mathbf{CRLB}_k = \sum_{l=1}^{2^k} \sum_{m=1}^{2^k} \mathbf{CRLB}_{k;l,m} \cdot \Pr\{S_{k;l,m}\} \tag{2.15}$$

where $\Pr\{S_{k;l,m}\}$ is given in (2.9).

3 A case study

In this section we apply the above mentioned theory to the tracking of a ballistic target; in particular, we assume that a radar is acquiring few measurements during the ballistic target flight: the problem is to estimate the impact point of the target [7].

To calculate the accuracy of the impact point, a simplified model of a ballistic target has been considered. The algorithm that has been conceived is based on processing the acquired measurements with a batch approach; it accounts for detection probability $P_d < 1$ and false alarm probability $P_{fa} > 0$.

The ballistic target has been modelled according to the following hypotheses:

- instantaneous thrust;
- flat Earth surface;
- gravity acceleration g constant with target height;
- no drag (point like target);
- radar measurements accuracy constant with target range;
- inertial coordinate reference system.

The time of launch is assumed to be known a priori; in this example, $t_0 = 0$. If the time of launch is unknown, it can be inserted as a component in the state vector to be estimated. It will then be assigned a large initial standard deviation that will affect the prediction performance. The trajectory equations are:

$$\begin{cases} x(t) = x_0 + v_x t \\ y(t) = y_0 + v_y t \\ z(t) = z_0 + v_z t - \frac{1}{2} g t^2 \end{cases} \tag{3.1}$$

where $\mathbf{x} = [x_0 \; y_0 \; z_0 \; v_x \; v_y \; v_z]^T$ is the vector parameter to estimate.

[2] The averaging of the measurements is a matter of the processing design: the processor may also select only one of the two measurements or average only one Cartesian/polar coordinate. For mathematical tractability, straight averaging is assumed.

The components of the unknown $\mathbf{x}$ are calculated via the MLE technique. The unbiased estimator $\hat{\mathbf{x}}$ is obtained by solving the minimisation problem $\hat{\mathbf{x}} = \underset{\mathbf{x}}{arg\,min}[\lambda(\mathbf{x})]$, where $\lambda(\mathbf{x})$ is the MLE functional:

$$\lambda(\mathbf{x}) = n\left[ln\left(\sigma_\rho\sqrt{2\pi}\right) + ln\left(\sigma_\phi\sqrt{2\pi}\right) + ln\left(\sigma_\vartheta\sqrt{2\pi}\right)\right] + \sum_{k=1}^{n}\left\{\begin{array}{l}\dfrac{[\rho(k)-h(\mathbf{x},k)]^2}{2\sigma_\rho^2} + \\ + \dfrac{[\phi(k)-f(\mathbf{x},k)]^2}{2\sigma_\phi^2} + \dfrac{[\vartheta(k)-l(\mathbf{x},k)]^2}{2\sigma_\vartheta^2}\end{array}\right\} \quad (3.2)$$

In Eq. (3.2), n is the number of scans, σ_ρ, σ_θ, σ_ϕ are the standard deviations of radar measurements: range ρ, azimuth θ, elevation ϕ. The first three terms, not depending on $\boldsymbol{x}$, do not contribute to the calculation of the unbiased estimator, so we can assume that the MLE functional consists only of the other three components, which instead depend on the vector parameter $\boldsymbol{x}$:

$$\begin{cases} h(\mathbf{x},k) = \sqrt{(x(kT))^2 + (y(kT))^2 + (z(kT))^2} \\ f(\mathbf{x},k) = arcsin\dfrac{z_0 + v_z\cdot kT - \frac{1}{2}g(kT)^2}{\sqrt{(x(kT))^2 + (y(kT))^2 + (z(kT))^2}} \\ l(\mathbf{x},k) = arctg\dfrac{x_0 + v_x\cdot kT}{y_0 + v_y\cdot kT} \end{cases} \quad (3.3)$$

where T is the radar measurement scan period.

Note that the mathematical expression of the MLE (Eq. (3.2)) is also applicable to the case of $P_d<1$ and $P_{fa}>0$. Each scan k is managed according to the following assumptions:

1. if the target is detected and the false alarm simultaneously occurs, the measurement vector $z(k) = [\rho(k), \theta(k), \phi(k)]$ is obtained by a simple average: $z(k) = 0.5\cdot[z_T(k) + z_F(k)]$, being $z_T(k)$ and $z_F(k)$ the measurement vectors respectively associated to the target and to the false alarm;
2. if only the target is detected, then $z(k) = z_T(k)$;
3. if only the false alarm occurs, then $z(k) = z_F(k)$;
4. if neither the target is detected nor the false alarm occurs, the measurement is missed and does not contribute to the MLE functional.

The CRLB sets a lower bound on the variance of any unbiased estimator, giving the best estimation accuracy for $n \to \infty$. The covariance matrix of the unbiased estimator $\hat{\mathbf{x}}$ is bounded by the inverse of the FIM:

$$\begin{array}{l} E\left\{(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^T\right\} \geq \mathbf{J}^{-1} \\ \mathbf{J}(\mathbf{x}) = E\left\{[\nabla_\mathbf{x}\lambda(\mathbf{x})][\nabla_\mathbf{x}\lambda(\mathbf{x})]^T\right\}_{\mathbf{x}=\hat{\mathbf{x}}} \end{array} \quad (3.4)$$

The expression of the FIM for our problem is:

$$\mathbf{J} = \sum_{k=1}^{n}\left\{\begin{array}{l} \dfrac{1}{\sigma_\rho^2}\nabla_\mathbf{x}h(\mathbf{x},k)(\nabla_\mathbf{x}h(\mathbf{x},k))^T + \\ + \dfrac{1}{\sigma_\phi^2}\nabla_\mathbf{x}f(\mathbf{x},k)(\nabla_\mathbf{x}f(\mathbf{x},k))^T + \\ + \dfrac{1}{\sigma_\vartheta^2}\nabla_\mathbf{x}l(\mathbf{x},k)(\nabla_\mathbf{x}l(\mathbf{x},k))^T \end{array}\right\} \quad (3.5)$$

A mathematical expression of some of the above terms can be found in [8].

The previous equations (3.4) and (3.5) apply to the classical case (P_d=1, P_{fa}=0). In the following, the theoretical CRLBs for the target and false alarms are calculated supposing that the detection probability P_d is not equal to unity and also considering the detection event given by a false alarm (i.e., P_{fa}>0).

Given a certain number n of scans, the equivalent CRLB has been calculated following the next four steps:

1. enumeration of the 2^{2n} possible sequences, obtained after combining the 2^n binary sequences relative to the target and the 2^n binary sequences relative to the false alarm;
2. calculation of the occurrence probability associated to each sequence;
3. evaluation of the CRLB relative to the specific sequence, according to Eq. (2.14);
4. sum of all the CRLBs calculated in step 3, after weighting each one by the corresponding occurrence probability evaluated in step 2, as expressed by Eq. (2.15).

A simple explanatory example relative to two scans (n=2) is illustrated in Table 1.

Target		**False alarm**			
1st scan	2nd scan	1st scan	2nd scan	Event probability	CRLB
1	1	1	1	$P_d^2\cdot P_F^2$	$\mathbf{CRLB}_{1111}$
1	1	1	0	$P_d^2\cdot P_F\cdot(1-P_F)$	$\mathbf{CRLB}_{1110}$
1	1	0	1	$P_d^2\cdot P_F\cdot(1-P_F)$	$\mathbf{CRLB}_{1101}$
1	1	0	0	$P_d^2\cdot(1-P_F)^2$	$\mathbf{CRLB}_{1100}$
1	0	1	1	$P_d\cdot(1-P_d)\cdot P_F^2$	$\mathbf{CRLB}_{1011}$
…	…	…	…	...	...
0	0	0	0	$(1-P_d)^2\cdot(1-P_F)^2$	$\mathbf{CRLB}_{0000}$

Table 1. Enumerated events, their probability and CRLB for the two scan case.

Each row of Table 1 represents an event: the corresponding sequence is reported in columns 1÷4, the event probability and its covariance matrix are respectively indicated in columns 5 and 6. The symbol $\mathbf{CRLB}_{xyzw}$ stands for the covariance matrix associated to the binary sequence "xywz". When a measurement is missed, either relative to the target or to the false alarm, the corresponding variance is set to a very large value. As the event "0000" is actually meaningless, it is not included in the computation of the final equivalent CRLB, which encounters all the covariance matrices listed above weighted by their corresponding event probability.

More specifically, the actual steps taken to calculate the global CRLB are:

1. computation of matrix **J** as expressed by Eq. (3.5) for each target sequence "xy" and false alarm sequence "wz";
2. inversion of matrix **J** and consequent evaluation of the CRLB for each target sequence "xy" and false alarm sequence "wz";
3. calculation of the CRLB for the combined sequence "xywz" according to Eq. (2.14);
4. estimation of the global CRLB as expressed by (2.15).

The CRLB for the estimated vector $\mathbf{x}$ of target parameters (i.e. the covariance matrix of the unbiased estimator $\hat{\mathbf{x}}$) is:

$$\mathbf{CRLB} = \begin{bmatrix} \sigma_{x_0}^2 & \sigma_{x_0 y_0} & \sigma_{x_0 z_0} & \sigma_{x_0 v_x} & \sigma_{x_0 v_y} & \sigma_{x_0 v_z} \\ \sigma_{x_0 y_0} & \sigma_{y_0}^2 & \sigma_{y_0 z_0} & \sigma_{y_0 v_x} & \sigma_{y_0 v_y} & \sigma_{y_0 v_z} \\ \sigma_{x_0 z_0} & \sigma_{y_0 z_0} & \sigma_{z_0}^2 & \sigma_{z_0 v_x} & \sigma_{z_0 v_y} & \sigma_{z_0 v_z} \\ \sigma_{x_0 v_x} & \sigma_{y_0 v_x} & \sigma_{z_0 v_x} & \sigma_{v_x}^2 & \sigma_{v_x v_y} & \sigma_{v_x v_z} \\ \sigma_{x_0 v_y} & \sigma_{y_0 v_y} & \sigma_{z_0 v_y} & \sigma_{v_x v_y} & \sigma_{v_y}^2 & \sigma_{v_y v_z} \\ \sigma_{x_0 v_z} & \sigma_{y_0 v_z} & \sigma_{z_0 v_z} & \sigma_{v_x v_z} & \sigma_{v_y v_z} & \sigma_{v_z}^2 \end{bmatrix} \tag{3.6}$$

Assume now that we wish to predict the target state ahead in time after the acquisition of n radar measurements; we are interested in the target position in kT instants with $k > n$. We achieve the purpose by inserting the target state vector $\hat{\mathbf{x}}$, estimated by the n radar measurements, in the target model (3.1) which will be calculated at the k^{th} time instant. The corresponding covariance matrix $\mathbf{P}$ of the predicted target position is:

$$\mathbf{P} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \tag{3.7}$$

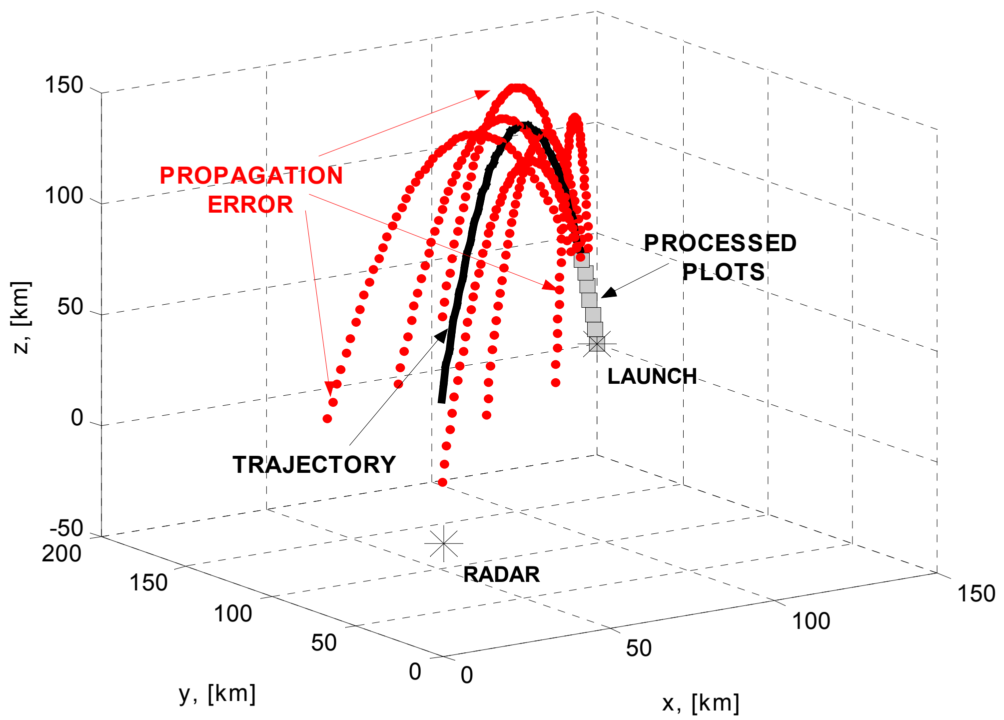

Fig. 1: Trajectory of a ballistic target (solid line), from launch to impact points. Processed plots (squares) and estimation uncertainty (•) are shown.

where

$$\begin{cases} \sigma_x^2(kT) = \sigma_{x_0}^2 + \sigma_{v_x}^2 \cdot (kT)^2 + 2\sigma_{x_0 v_x} \cdot (kT) \\ \sigma_y^2(kT) = \sigma_{y_0}^2 + \sigma_{v_y}^2 \cdot (kT)^2 + 2\sigma_{y_0 v_y} \cdot (kT) \\ \sigma_z^2(kT) = \sigma_{z_0}^2 + \sigma_{v_z}^2 \cdot (kT)^2 + 2\sigma_{z_0 v_z} \cdot (kT) \\ \sigma_{xy}(kT) = \sigma_{x_0 y_0} + \left(\sigma_{y_0 v_x} + \sigma_{x_0 v_y}\right) \cdot (kT) + \sigma_{v_x v_y} \cdot (kT)^2 \\ \sigma_{xz}(kT) = \sigma_{x_0 z_0} + \left(\sigma_{z_0 v_x} + \sigma_{x_0 v_z}\right) \cdot (kT) + \sigma_{v_x v_z} \cdot (kT)^2 \\ \sigma_{yz}(kT) = \sigma_{y_0 z_0} + \left(\sigma_{y_0 v_z} + \sigma_{z_0 v_y}\right) \cdot (kT) + \sigma_{v_y v_z} \cdot (kT)^2 \end{cases} \tag{3.8}$$

The variances and co-variances in the equation above are given by (3.6).

This algorithm has been applied to the ballistic target whose trajectory (range = 80 km, maximum height = 110 km) is shown in Fig. 1.

The radar data used in the calculation are:

- scan period T = 5 s;
- range measurement standard deviation σ_ρ = 100 m;
- azimuth measurement standard deviation σ_θ = 0.3°;
- elevation measurement standard deviation σ_ϕ = 0.3°.

The results obtained after running a suitable software tool, with P_d, P_F and the number of scans as variables, are reported in Table 2.

P_d	P_F	Number of scans	Impact σ_x [km]	Impact σ_y [km]	Impact area [km^2]
1	0	8	9.0	6.8	51.2
0.9	0	8	9.3	6.9	53.6
0.9	0.01	8	9.3	6.9	53.8
0.9	0.05	8	10.4	7.8	68.2
1	0	7	11.2	8.4	73.0
0.9	0	7	11.5	8.6	77.4
0.9	0.01	7	11.6	8.7	78.0
0.9	0.05	7	13.3	10.0	104.1
1	0	6	14.3	10.7	110.2
0.9	0	6	14.9	11.2	119.4
0.9	0.01	6	14.9	11.2	120.0
0.9	0.05	6	16.9	12.7	154.0

Table 2: CRLB values calculated with the enumeration method for variable P_d, P_F and number of scans.

The input parameters (P_d, P_F and the number of scans) are reported in columns 1÷3; the results (impact point σ_x, σ_y and the corresponding uncertainty area A) are shown in columns 4÷6. The uncertainty area A is:

$$A = \pi \sqrt{\sigma_x^2 \cdot \sigma_y^2 - \sigma_{xy}^2} \tag{3.9}$$

We note that the impact accuracy diminishes with the decrease of the number of scans and of detection probability P_d and with the increase of false alarm probability P_F. In particular, when the number of scans is 8, the impact area worsens by 33%, comparing the least favorable case (P_d=0.9, P_F=0.05) to the best one (P_d=1, P_F=0). The smaller the probability of false alarm per scan P_F, the less significant its influence on the estimation accuracy: when $P_F=10^{-4}$, false alarms do not affect the impact point prediction. Of course, the accuracy strongly reduces with the increase of the radar measurement scan period *T* (this study case is not explicitly reported here).

4 Monte Carlo simulation results

A Monte Carlo simulation has been run to study the performance of the ML estimator given by Eq. (3.2) and validate the expression (2.14) we found for the CRLB.

The simulation refers to the case study described in Section 3 and has been organized as follows:

1. ideal trajectory generation out of the Monte Carlo loop;
2. generation of a random variable in the [0,1] range to assess the detection or missed detection of the k^{th} target measurement (out of *n*) in the specific Monte Carlo run;
3. in case of target detection, generation of measurement noise and its addition to the real trajectory; in case of missed detection, no measurement is generated;
4. generation of a random variable in the [0,1] range to assess either the presence or the absence of the k^{th} false alarm measurement (out of *n*) in the specific Monte Carlo run;
5. in case of false alarm presence, generation of three times the measurement noise and its addition to the real trajectory; in case there is no false alarm, no measurement is generated;
6. if target is detected and false alarm occurs, we simply average the two measurements; if one of the two measurements (target or false alarm) is missed, the other one is processed; if no measurement is acquired, the plot is missed;
7. at the end of the generic Monte Carlo run, calculation and minimization of the MLE functional to estimate the vector parameter **x**;
8. at the end of the Monte Carlo loop, calculation of the bias and of the covariance matrix of the estimated **x**.

The results of the Monte Carlo simulation, after running 2000 trials, are shown in Table 3. The input parameters (P_d, P_F and the number of scans) are reported in columns 1÷3; the results (impact point σ_x, σ_y and the corresponding uncertainty area A) are shown in columns 4÷6.

P_d	P_F	Number of scans	Impact σ_x [km]	Impact σ_Y [km]	Impact area [km^2]
1	0	8	9.0	6.8	52.5
0.9	0	8	9.9	7.4	61.3
0.9	0.01	8	9.9	7.4	63.0
0.9	0.05	8	10.5	7.8	70.0
1	0	7	11.3	8.5	73.7
0.9	0	7	12.2	9.1	88.6
0.9	0.01	7	12.7	9.4	93.0
0.9	0.05	7	14.2	10.6	111.1
1	0	6	14.8	11.0	114.3
0.9	0	6	16.2	12.1	140.8
0.9	0.01	6	16.5	12.3	141.7
0.9	0.05	6	17.7	13.3	164.4

Table 3: Monte Carlo simulation results (2000 trials) for variable P_d, P_F and number of scans.

Monte Carlo simulation results of Table 3 should be compared to the theoretical CRLB values of Table 2. Note that they are all greater but close to the corresponding values obtained by means of the enumeration method applied to the CRLB, thus confirming the accuracy of expression (2.14).

5 Conclusions

This paper has extended the classical theory of parametric estimation of MLE to the case in which the measurements are acquired with $P_d<1$ and also in presence of potential false alarms (i.e. $P_{fa}>0$). The analysis was built around the assumption of having just one false alarm. A suitable expression of the CRLB has been found and applied to the estimation study of a ballistic target flight. A Monte Carlo simulation has successfully validated the new theoretical CRLB enumeration method. Our future work will be aimed at applying the enumeration technique to the recursive filtering case for $P_d<1$ and $P_{fa}>0$.

6 References

[1] S. Kay, "Fundamentals of Statistical Signal Processing. Estimation Theory", Prentice Hall Signal Processing Theory, Upper Saddle River, NJ 07458, USA, 1993.

[2] C. Jauffret, and Y. Bar-Shalom, “Track Formation With Bearings and Frequency Measurements in Clutter”, IEEE Trans. on Aerospace and Electronic Systems, vol. 26, no. 6, pp. 999-1010, 1990.

[3] T. Kirubarajan and Y. Bar-Shalom, "Low Observable Target Motion Analysis Using Amplitude Information", IEEE Trans. on Aerospace and Electronic Systems, vol. 32, no. 4, pp. 1367-1384, 1996.

[4] M.L. Hernandez, A.D. Marrs, N.J. Gordon, S. Maskell, and C.M. Reed, "Cramér-Rao Bounds for Non-Linear Filtering with Measurement Origin Uncertainty", Proc. Of the 5th Intl. Conf. On Info Fusion, vol. 1, pp. 18-25, 2002.

[5] A. Farina, B. Ristic, and L. Timmoneri, "Cramér-Rao Bound for Nonlinear Filtering With P_d <1 and Its Application to Target Tracking", IEEE Trans. Signal Processing, vol. 50, no. 8, pp. 1916-1924, August 2002.

[6] M. Hernandez, B. Ristic, A. Farina, and L. Timmoneri, "Comparison of Two Posterior Cramér-Rao Bounds for Non-Linear Filtering with Applications", IEEE Trans. Signal Processing, vol. 52, no. 9, pp. 2361-2370, September 2004.

[7] A. Farina, D. Benvenuti, and B. Ristic, "Estimation Accuracy of a Landing Point of a Ballistic Target", Proc. Of Intl. Conf. Fusion, Washington DC, pp. 2-9, May 2002.

[8] A. Farina, "Target Tracking with Bearings – Only Measurements", Signal Processing, vol. 78, no. 1, pp. 61-78, October 1999.