

CRLB with $P_d < 1$ fused tracks

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***Abstract** - This paper presents mathematical procedures to calculate the Cramer-Rao Lower Bound (CRLB) for a target track obtained by fusing two tracks given by two Interactive Multiple Model (IMM) trackers filtering the plots received by two radars. The target under tracking is a ballistic target (BT) in reentry phase and it is assumed that the two sensors could have a detection probabilities P_d less than 1. Accurate BT tracking is required for: (i) impact point prediction, (ii) support to the kill assessment, and (iii) threat classification. Note that, to improve the performance, the method can be extended to the fusion of more than two sensor tracks. The paper is organized as follows: the problem is introduced in section 1; section 2 briefly recalls the models of BT dynamic and of the radar measures. Section 3 presents the tracking algorithm based on the IMM approach; section 4 contains the description of the CRLB evaluation problem and section 5 the results achieved in comparing the fusion of two BT tracks (each track is obtained from the IMM algorithm described in section 4) with the CRLB; Monte Carlo simulations is also used to quantify the fused track accuracies. Section 6 is pertinent to conclusions and follows on of the activity while references are reported in section 7.*

1 Introduction

The problem of BT tracking has been described in the specialized technical literature, see for instance [1-3]. This paper deals with the problems of:

- fusing the tracks of two radar sensors,
- computing the CRLB of the target position and velocity accuracies achievable by fusing the two tracks when the detection probability of the sensors is less than 1;

- comparing the CRLB with the performance achievable with two tracks computed via an IMM-UKF (Unscented Kalman Filter) tracking algorithm.

The Cramer-Rao bounds for nonlinear stochastic filtering [4, 5] some times referred to as Posterior CRLB-PCRLB) have been derived with an implicit assumption that the measurement sensor is operating with the probability of detection, P_d , equal to unity. Since in many applications - such as target tracking - this is not the case, in this paper we consider the theoretical CRLBs for the case where $P_d < 1$. The detection event given by a false alarm is not considered because the probability of false alarm is much smaller than the detection probability (in a typical radar system $P_d = 0.9$ and $P_{fa} = 10^{-6}$). Two methodologies, Enumeration and IRF (Information Reduction Factor), have been widely described in [7, 8, 13] for the computation of CRLB when $P_d < 1$.

The enumeration solution for the CRLB when $P_d < 1$ is based on the evaluation of an exponentially growing number of possible miss/detection sequences. As the number of sensor scans grows, it becomes more difficult (practically impossible) to compute the exact theoretical bound. Therefore in [7] a practically feasible approximation has been proposed and verified by simulations in the linear and non-linear stochastic filtering cases. In this paper the enumeration method is applied to the CRLB of two fused tracks; the achieved results are the subject of section 5.

2 Models of dynamic of BT and radar measures

Three main forces affect the BT motion: thrust, drag and gravity. For the sake of this paper, it is assumed that the BT is in the cruise phase during the BT state vector estimation while drag and gravity are acting on the target body during the re-entry phase. The drag acceleration expression is [1-3].

$$\mathbf{a}_{drag} = \begin{bmatrix} a_{dragx} \\ a_{dragy} \\ a_{dragz} \end{bmatrix} = -\frac{1}{2} \frac{\rho(z) \cdot g_0}{\beta} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (2.1)$$

where β is the ballistic coefficient (N/m²), $\rho(z)$ is the air density function of the height:

$$\rho(z) = 1.21907 \cdot e^{-z/9146.64} \quad (2.2)$$

$\dot{x}, \dot{y}, \dot{z}$ are the velocity components of the BT along the three axes of a Cartesian reference system. The gravity acceleration is considered constant, $g_0=9.8\text{m/s}^2$; the gravity acceleration is expressed as.

$$\mathbf{a}_g = \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix} \quad (2.3)$$

The measurements, collected by the radar for target tracking, are the range r , elevation ε and azimuth ϑ .

The error standard deviations of these measurements are denoted as σ_r (for range), σ_ε (for elevation) and σ_ϑ (for azimuth). Radar measurements are transformed to the Cartesian coordinates so that the measurement equation is linear

$$\mathbf{z}_k = \mathbf{H}\mathbf{s}_k + \mathbf{v}_k \quad (2.4)$$

where \mathbf{v}_k is the noise on the measured Cartesian co-ordinates; it is zero-mean white Gaussian with covariance matrix \mathbf{R}_k . For all practical purposes this is a good approximation, which greatly simplifies the tracking algorithm; otherwise one would also have to take into consideration the non-linearity of the measurement equation.

3 IMM-UKF architecture

The theory and the application of the IMM have been the subject of many publications, see for instance [9]. The rationale for choosing the IMM approach for the tracking of a potential BT is essentially due to the fact that the BT characteristics are not generally “a priori” known, thus it is required to “on-line” estimate the BT parameters to maximize the tracker accuracy. The IMM offers the possibility of mixing the output of different filters designed for different BTs, each one having the possibility of adapting its parameters to the target to be tracked, thus permitting the correct tracking of BTs pertaining to different “classes”. In addition to this, the probability of selecting one of the filters

existing in the bank of the IMM gives a clear indication of the confidence of the tracker on the type of target under analysis; this is an intrinsic capability of non co-operative target classification, available “for free” by the IMM. Each filter of the IMM proposed in this paper is an Unscented Kalman Filter (UKF) [10] designed taking into account the BT dynamic equations described in section 2. The UKF is a recursive MMSE (Minimum Mean Square Error) estimator that does not approximate the non-linear state and measurement equations. It uses the true non-linear model of state and/or measurements equation but approximates the pdf of the state vector. This density is still Gaussian, but is specified by a set of deterministically chosen samples (or *sigma* points). The sigma points completely capture the true mean and covariance of the Gaussian density and when propagated through the non-linear system, capture the posterior mean and covariance accurately to the second order for any non-linearity.

The unscented transform (UT) is a method for calculating the statistics of a random vector, which undergoes a non-linear transformation. Let \mathbf{x} be the n_x dimensional random vector,

$\mathbf{g} : \mathbf{R}^{n_x} \rightarrow \mathbf{R}^{n_y}$ a non-linear function and $\mathbf{y} = \mathbf{g}(\mathbf{x})$.

Assume the mean and the covariance of \mathbf{x} are $\bar{\mathbf{x}}$ and \mathbf{P}_x respectively. The simple procedure for the calculation of the first two moments of \mathbf{y} using the UT is as follows [6].

1. Compute $2n_x$ sigma points χ_i and their weights W_i

$$\chi_i = \bar{\mathbf{x}} + \left(\sqrt{n_x \mathbf{P}_x} \right)_i \quad (3.1)$$

$$i = 1, \dots, n_x$$

$$W_i = 1/(2n_x) \quad (3.2)$$

$$\chi_i = \bar{\mathbf{x}} - \left(\sqrt{n_x \mathbf{P}_x} \right)_i \quad (3.3)$$

$$i = n_x + 1, \dots, 2n_x$$

$$W_i = 1/(2n_x) \quad (3.4)$$

where $\left(\sqrt{(n_x \mathbf{P}_x)} \right)_i$ is the i -th row or column of the matrix

square root of $n_x \mathbf{P}_x$. The weights are normalised (i.e. add up to 1).

2. Propagate each sigma point through the non linear function

$$\mathbf{y}_i = \mathbf{g}(\chi_i) \quad i = 0, \dots, 2n_x \quad (3.5)$$

Estimated mean and covariance of \mathbf{y} are computed as

$$\begin{aligned}\bar{\mathbf{y}} &= \sum_{i=1}^{2n_x} W_i y_i \\ \mathbf{P}_y &= \sum_{i=1}^{2n_x} W_i (y_i - \bar{\mathbf{y}})(y_i - \bar{\mathbf{y}})^T\end{aligned}\quad (3.6)$$

Next we describe the implementation of the UKF assuming that at time k the state estimate and its covariance are $\mathbf{s}_{k|k}$ and $\mathbf{P}_{k|k}$ respectively.

- Compute sigma points $\xi_{k|k}(i)$ and weights W_i ($i=1, \dots, 12$) corresponding to $\mathbf{s}_{k|k}$ and $\mathbf{P}_{k|k}$;
- Propagate sigma points using state equation (described in the previous section) as follows

$$\xi_{k+1|k}(i) = \Phi_k[\xi_{k|k}(i)] + \mathbf{G}(\mathbf{a}_g + \mathbf{a}_{\alpha\beta}) \quad (3.7)$$

- \mathbf{G} is a 7×3 matrix where T is the radar scan time:

$$\mathbf{G} = \begin{bmatrix} T^2/2 & T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T^2/2 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T^2/2 & T & 0 \end{bmatrix}^T \quad (3.8)$$

- Compute the mean and covariance of the predicted state $\mathbf{s}_{k+1|k}$ and $\mathbf{P}_{k+1|k}$ using predicted sigma points $\xi_{k+1|k}(i)$, weights W_i as follows

$$\begin{aligned}\hat{\mathbf{s}}_{k+1|k} &= \sum_{i=0}^8 W_i \xi_{k+1|k}(i) \\ \mathbf{P}_{k+1|k} &= \mathbf{Q} + \sum_{i=0}^8 W_i [\xi_{k+1|k}(i) - \hat{\mathbf{s}}_{k+1|k}] [\xi_{k+1|k}(i) - \hat{\mathbf{s}}_{k+1|k}]^T\end{aligned}\quad (3.9)$$

where \mathbf{Q} is the process noise non singular covariance matrix:

$$\mathbf{Q} = q \begin{pmatrix} \Psi & 0 & 0 & 0 \\ 0 & \Psi & 0 & 0 \\ 0 & 0 & \Psi & 0 \\ 0 & 0 & 0 & \sigma_\beta^2 \end{pmatrix} \quad (3.10)$$

with

$$\Psi = \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix} \quad (3.11)$$

and q is a scalar quantity accounting for the uncertainty on the target model; the variance σ_β^2 expresses the uncertainty on the ballistic coefficient.

- Predict measurements derived from sigma points, that is

$$\mathfrak{s}_{k+1|k}(i) = \mathbf{H}\xi_{k+1|k}(i) \quad (3.12)$$

- Predict measurement and covariances

$$\begin{aligned}\hat{\mathbf{z}}_{k+1|k} &= \sum_{i=0}^8 W_i \mathfrak{s}_{k+1|k}(i) \\ \mathbf{P}_{zz} &= \mathbf{R}_{k+1} + \sum_{i=0}^8 W_i [\mathfrak{s}_{k+1|k}(i) - \hat{\mathbf{z}}_{k+1|k}] [\mathfrak{s}_{k+1|k}(i) - \hat{\mathbf{z}}_{k+1|k}]^T \\ \mathbf{P}_{sz} &= \sum_{i=0}^8 W_i [\xi_{k+1|k}(i) - \hat{\mathbf{s}}_{k+1|k}] [\mathfrak{s}_{k+1|k}(i) - \hat{\mathbf{z}}_{k+1|k}]^T\end{aligned}\quad (3.13)$$

where \mathbf{P}_{zz} , \mathbf{P}_{sz} are, respectively, the covariance matrices of the measurement and the cross-covariance of the measurement and the state variable.

- Compute the UKF gain and update state and covariance

$$\begin{aligned}\mathbf{K}_{k+1} &= \mathbf{P}_{sz} \mathbf{P}_{zz}^{-1} \\ \mathbf{s}_{k+1|k+1} &= \mathbf{s}_{k+1|k} + \mathbf{K}_{k+1} (\mathbf{z}_{k+1} - \hat{\mathbf{z}}_{k+1|k}) \\ \mathbf{P}_{k+1|k+1} &= \mathbf{P}_{k+1|k} - \mathbf{K}_{k+1} \mathbf{P}_{zz} \mathbf{K}_{k+1}^T\end{aligned}\quad (3.14)$$

4 Evaluation of CRLB

4.1 CRLB computation for one-sensor tracker with $P_d=1$

Tracking a BT is difficult because the motion equations are highly non-linear due to forces like drag and thrust. An optimum-tracking filter cannot be found in principle; thus sub-optimum filters are the usual current practice. The UKF is one of these. One open problem is to determine which is the best sub-optimum tracking filter. In terms of performance this can be done by resorting to recent theoretical findings in the field of CRLB for discrete-time non-linear filtering [4,5]; remarkably this theory is applicable also when the probability of detection of a sensor is less than unity [7, 8, 13].

The CRLB, which depends on the measurement model, sensor characteristics and the target state model, plays an important role in algorithm evaluation and assessment of the level of approximation introduced by a particular tracking filter. For a general non-linear filtering problem, the optimal recursive state estimator (the tracker) in the Bayesian sense requires the complete posterior density of the state to be

determined as a function of time. In the special case of linear/Gaussian estimation, the required density is Gaussian and the solution is the well-known Kalman filter, which gives also the accuracy of the estimated track. In the general non linear/non-Gaussian case the problem is very difficult and has no analytic closed form solution. A theoretical formula has been found in [7, 8]. The above mentioned references give the strategy to include in the CRLB computation a detection probability <1 .

The findings of [4, 7, 8] allow the CRLB computation for the BT tracking accuracy by means of the following methodology:

- (i) compute the trajectory for the BT using the equations of section 3; process noise is assumed to be absent;
- (ii) derive the Extended Kalman Filter (EKF) for BT as described in [2, 3]; suppose the a-priori knowledge of the BT parameters (drag coefficient as an example);
- (iii) run the EKF of step (ii) using the “nominal” trajectory of step (i) considering the radar measurement errors only in the definition of the \mathbf{R} matrix. In other word, no Monte Carlo trials are required being the effect of the radar uncertainty considered in the computation of the EKF gain matrix.

The tracker accuracy found with the above mentioned procedures are the CRLB for the BT trajectory under analysis. Once that the CRLB is obtained, it is possible to compare the performance of any other tracking architecture; the comparison with the IMM-UKF that we have developed is the subject of next section 6.

4.2 CRLB computation and IMM-UKF tracks fusion for $P_d=1$

A number of strategies have been conceived for fusing information (plots and/or tracks) generated by N sensors. In the case of N tracks, they are combined in order to achieve a single multisensor track for each target. This is performed after having associated the corresponding tracks by resorting to a statistical test [11, 13]. If the tracks are independents, they are merged into a single equivalent track as follows (see [11, page 17]).

$$\hat{\mathbf{s}} = \hat{\mathbf{P}} \cdot \sum_{i=1}^N [\hat{\mathbf{P}}_i]^{-1} \hat{\mathbf{s}}_i \quad (4.1)$$

$$\hat{\mathbf{P}} = \left\{ \sum_{i=1}^N [\hat{\mathbf{P}}_i]^{-1} \right\}^{-1} \quad (4.2)$$

where:

$\hat{\mathbf{S}}$ = state vector of multisensor track,

$\hat{\mathbf{P}}$ = covariance matrix of $\hat{\mathbf{S}}$,

$\hat{\mathbf{S}}_i$ = state vector of i -th monosensor track,

$\hat{\mathbf{P}}_i$ = covariance matrix of $\hat{\mathbf{S}}_i$.

A schematic drawing of this fusion algorithm is shown in figure 1 for $N=2$, where $(\theta_m)_{1,2}$ represents the target measurements from the two sensors.

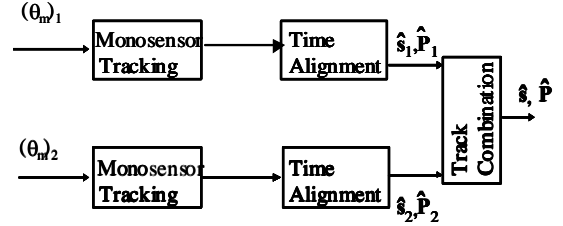


Figure 1: Multi-sensor filtering through combination of tracks.

In particular, equation (4.2) provides the CRLB of the fused track if the procedure of section 4.1 is applied; note that the same equation (4.2) is valid and applied to the fusion of two IMM-UKF tracks. In the case in which the tracks are dependent, the fusion algorithm is the one described in [12]. In the following we will consider only the case of independent tracks.

In this paper we hypothesize that the CRLB of independent tracks can be combined via eq. (4.2) to provide the CRLB of the fused track. This hypothesis is not demonstrated here and needs further consideration in future studies.

4.3 CRLB computation and IMM-UKF tracks fusion for $P_d < 1$

This section is dedicated to the computation of the CRLB in case of two sensors which have their own $P_{d_i} < 1$, $i=1,2$. To this aim the following two steps are required:

1. the covariance matrix \mathbf{P}_i of the track provided by i -th sensor is modified to account for the sensor $P_{d_i} < 1$, this is done by resorting to the enumeration method [7, 8, 13];
2. equation (4.2) is calculated by inserting \mathbf{P}_i determined under the condition of $P_{d_i} < 1$, $i=1,2$.

The CRLB is obtained if the procedure of section 4.1 is followed; the same procedure applies to the IMM-UKF study case.

5 Achieved Results

Purpose of this section is to show the benefit given by the fusion of two tracks pertaining to a BT in cruise and re-entry phases tracked by the two radars. The CRLB of single radar and two radars has been evaluated to quantify the advantages offered by the track fusion. Two study cases has been

considered: (i) $P_d=1$ as described in section 4.1 and 4.2, (ii) $P_d<1$ as presented in section 4.3. This last case represents a more realistic radar scenario requiring the use of enumeration method to evaluate the CRLB. It will be shown that a tracker based on IMM-UKF (see section 3) gives performance close to the CRLB both in the case of $P_d=1$ and $P_d<1$. The match of IMM-UKF accuracy with the CRLB also in the case of $P_d<1$ is an additional confirmation of the validity of the enumeration approach.

The study case selected for the CRLB & IMM-UKF comparison (single and double radars) concerns a simulated BT with following characteristics: single stage, linear consumption of propellant vs. flight time, drag coefficient=80000 N/m², radar detection range=150 km, target radar cross section=1 m² in approaching flight. Two radars are considered in the simulated scenario at a distance of 5 km; the two radars are named rad1 and rad2 in the following. The parameters of these two notional radars are: range accuracy=25m, azimuth accuracy=0.1°, elevation accuracy =0.1°, data rate=6 seconds, either $P_d=1$ or $P_d=0.9$. The IMM is constituted by three UKFs with different initial values of β (10000, 40000 and 300000 N/m²). To avoid a large number of figures, it has been decided to plot the cubic root of volume of uncertainty (one sigma) ellipsoid of the target track.

Figure 2 depicts the relationship between the filter accuracy and CRLB for the single radar and the two values of P_d .

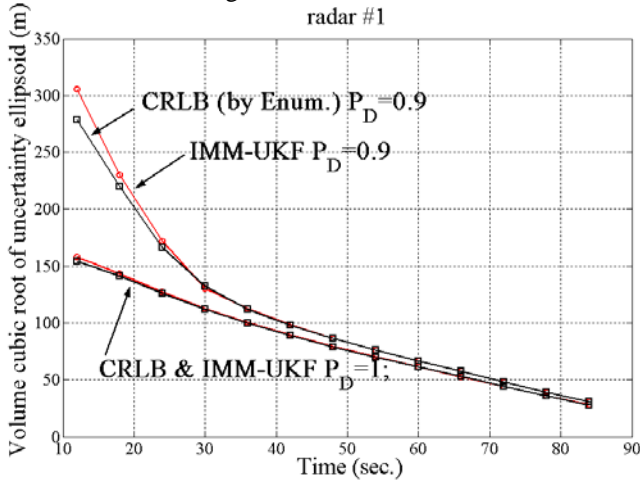


Figure 2: Comparison of CRLB and IMM-UKF track for single radar and $P_d=1$ and 0.9.

Note the good matching between CRLB calculated with enumeration and IMM-UKF also in the case of $P_d=0.9$.

Figure 3 reports the CRLB ($P_d=0.9$) for the following cases: (i) single radar, data rate 6 sec (both rad1 and rad2), (ii) fused track, (iii) an equivalent single radar placed in the mid point between rad1 and rad2 operating with a data rate of 3 seconds.

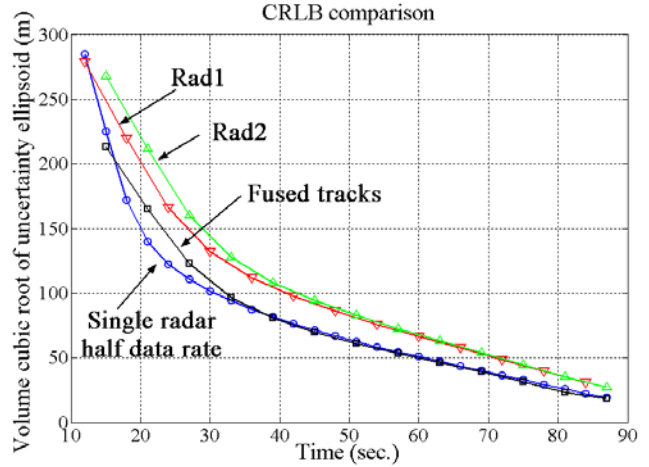


Figure 3: Comparison of CRLB for single radar and two radars with $P_d=0.9$.

It is possible to observe the improvement obtained by the fusion of the two tracks with respect to the track of each single radar, giving accuracy practically coincident with the track of the equivalent radar with a half data rate.

Finally, in figure 4, the IMM-UKF fused tracks are compared with the corresponding CRLB in both case $P_d=1$ and $P_d=0.9$.

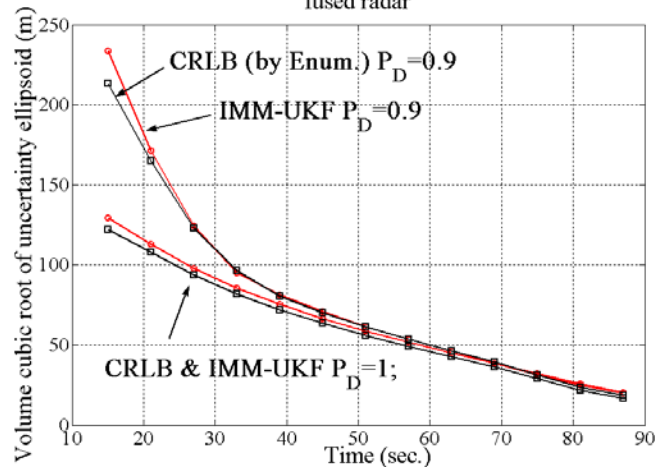


Figure 4: Comparison of CRLB and IMM-UKF tracks for two radars (fusion data) and $P_d=1$ and 0.9.

From figure 4 we note the almost ideal performance of the IMM-UKF.

6 Conclusions

The CRLB in case of fusing independent tracks pertaining to two radar sensors have been computed for the case of $P_d<1$. The enumeration method has been introduced and used for the analysis. The tracks are pertinent to a BT and a tracking filter based on an IMM-UKF has been compared with the CRLB. The benefit in terms of improvement of the tracker accuracy after the fusion has been clearly demonstrated.

7 References

1. D. Benvenuti, A. Farina, B. Ristic, "Estimation accuracy of a landing point of a ballistic target", Proc. of Intl. Conf. Fusion 2002, Washington DC, May 2002, pp. 2-9.
2. A. Farina, M.G. Del Gaudio, U. D'Elia, S. Immediata, L. Ortenzi, L. Timmoneri, M.R. Toma, "Detection and Tracking of Ballistic Target", 2004 IEEE Radar Conference, April 2004, Philadelphia (USA), pp. 450-456.
3. A. Farina, S. Immediata, L. Ortenzi, L. Timmoneri, "Tracking of ballistic target: comparison of an IMM architecture with the CRLB" Multinational Ballistic Missile Defense Conference, 21-24 July 2004, Berlin.
4. J.H. Taylor, "Cramer-Rao estimation error lower bound analysis for non linear systems with unknown deterministic variables". IEEE Trans. Automat. Contr., vol. AC 24, pp.343-344, Apr. 1979.
5. P. Tichavsky, C. H. Muravchik and A. Nehorai, "Posterior Cramer-Rao bounds for discrete-time nonlinear filtering", IEEE Trans. Signal Processing, vol. 46, pp.1386-1396. May 1998.
6. A. Farina, B. Ristic, D. Benvenuti, "Tracking a ballistic target: comparison of several non linear filters", IEEE Trans on AES, vol. 38, issue no. 3, July 2002, pp. 854-867.
7. A. Farina, B. Ristic, L. Timmoneri, "Cramer-Rao Bound for non linear filtering with $P_d < 1$ and its application to target tracking", IEEE Trans. on Signal Processing, vol. 59, no. 8 August 2002, pp 1916-1924.
8. M. Hernandez, B. Ristic, A. Farina, L. Timmoneri, "Comparison of two Cramer-Rao bounds for non linear filtering with $P_d < 1$ ", vol.52, no. 9, September 2004, pp. 2361-2370.
9. Y. Bar-Shalom, X. R. Li, "Estimation and Tracking: principles, techniques and software", Artech House, 1993.
10. S. J. Julier, J. K. Uhlmann, "Unscented Filtering and Nonlinear Estimation", Proc. of the IEEE, vol. 92, No. 3, March 2004, pp. 401-422.
11. A. Farina, F. A. Studer "Radar Data Processing, New York, Wiley, May 1986, Vol. II.
12. Y. Bar-Shalom, L. Campo, "The effect of the common process noise on the two sensors fused track covariance", IEEE Trans. on Aerospace and Electronic Systems, vol. 22, November 1986, pp. 803-805.
13. A. Farina, B. Ristic, S. Immediata, L. Timmoneri, "CRLB with $P_d < 1$ for bearings only fused tracks", Fusion 2003, Proceedings of the Sixth International Conference on International Fusion, July 8-11, 2003, vol. 2, pp. 919 - 925